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ON CERTAIN ENERGIES OF A ONE-POINT UNION OF COMPLETE GRAPHS K_n

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Abstract. The eigenvalues of a graph G are the eigenvalues of its adjacency matrix. The energy of the graph is defined as the sum of the absolute values of all its eigenvalues. In this paper we compute different energies of a one-point union of m copies of complete graphs on n vertices.

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1. Introduction

Let $G(V, E)$ be a simple connected undirected graph with p vertices and q edges. The adjacency matrix of G is the $0-1$ matrix $A = (A_{xy})$, where $A_{xy} = 1$ when there is an edge between vertices x and y in G and $A_{xy} = 0$, otherwise.

The characteristic polynomial of G is the characteristic polynomial of the adjacency matrix A and is denoted by $p_G(\lambda)$. The eigenvalues of G are the zeros of the characteristic polynomial and the *spectrum* of G is the multiset of eigenvalues of G denoted by $Spec(G)$. We write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ m_1 & m_2 & \cdots & m_p \end{pmatrix}$$

where $\lambda_1, \lambda_2 \cdots \lambda_p$ are the eigenvalues and m_i is the multiplicity of λ_i, $1 \leq i \leq p$. Unless we indicate otherwise, we shall assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. The largest eigenvalue λ_1 is called the *index* of G. The terminology and definitions that we adopt are as in [5, 8, 9].

Since the adjacency matrix A is real and symmetric, all its eigenvalues are real. Since A has zero diagonal, the trace of A and hence the sum of eigenvalues is zero. If $\lambda_1, \lambda_2 \ldots \lambda_p$ are the eigenvalues of G, then the *energy* of G is defined by

$$E(G) = \sum_{j=1}^{p} |\lambda_j| \tag{1}$$

The Laplace matrix of G is the matrix $L = (L_{xy})$ where $L_{xy} = D_{xy} - A_{xy}$ for $x \neq y$. If $D = (D_{xy})$ is the diagonal matrix such that D_{xx} is the degree of x, then $L = D - A$. The Laplace spectrum is the spectrum of the Laplace matrix. L is real and symmetric, so that the Laplace spectrum is real. If $\mu_1, \mu_2 \ldots \mu_p$ are the eigenvalues of the Laplace matrix, then the Laplacian

energy, denoted by $LE(G)$, is defined by

$$LE(G) = \sum_{i=1}^{p} |\mu_i - \frac{2q}{p}| \tag{2}$$

The matrix $Q = D + A$ is called the Q-Laplace matrix or the signless Laplace matrix of G. If $t_1 \geq t_2 \geq \ldots \geq t_p \geq 0$ are the eigenvalues of the Q-Laplace matrix of G then the Q-Laplacian energy of G is defined by

$$QE(G) = \sum_{i=1}^{p} |t_i - \frac{2q}{p}| \tag{3}$$

The Seidel matrix $S(G) = (s_{ij})$, is a modified adjacency matrix, defined by

$$s_{ij} = \begin{cases} -1, & \text{if } i \text{ and } j \text{ are adjacent } (i \neq j) \\ 1, & \text{if } i \text{ and } j \text{ are non-adjacent } (i \neq j) \\ 0, & i = j. \end{cases}$$

Then the Seidel energy is defined by

$$SE(G) = \sum_{j=1}^{p} |s_j| \tag{4}$$

where s_j are the eigenvalues of S.

The maximum degree matrix of order n is a matrix [1], whose (i, j)-element is

$$MD_{ij} = \begin{cases} max\{d_i, d_j\}, & \text{if } i \text{ and } j \text{ are adjacent } (i \neq j) \\ 0, & otherwise \end{cases}$$

where d_i is the degree of vertex i.

Then the maximum degree energy is defined by

$$MDE(G) = \sum_{j=1}^{p} |\sigma_j| \tag{5}$$

where σ_j are the eigenvalues of maximum degree matrix of G.

A graph G in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph. The graph $G^{(m)}$ obtained by identifying the roots of m copies of G is called the one-point union [7].

Eigenvalue calculation without the aid of a computing software is rarely easy. In this paper we consider the one-point union of m copies of the complete graph K_n which we denote by $K_n^{(m)}$. We compute different energies of $K_n^{(m)}$.

2. A few applications

The aim of this section is to draw the attention of the mathematical community to rapidly growing applications of the theory of graph spectra. Besides classical and well documented applications to Chemistry and Physics, there are applications of graph eigenvalues in Computer Science in various investigations. There are also applications in several other fields like Biology, Geography, Economics and Social Sciences.

Graph spectra and energy have several important applications in computer science. Graph spectra appear in internet technologies, pattern recognition, computer vision and in many other areas [4]. One of the oldest applications (from 1970's) of graph eigenvalues in computer science is related to graphs called expanders. The recent progress on expander graphs and eigenvalues was initiated by problems in communication networks. Expanders can be constructed from graphs with a small second largest eigenvalue in modulus. Such class of graphs includes the so called Ramanujan graphs [6].

The largest eigenvalue λ_1 plays an important role in modelling virus propagation in computer networks. The smaller the largest eigenvalue, the larger the robustness of a network against the spread of viruses. In fact, it was shown in [14] that the epidemic threshold in spreading viruses is proportional to $\frac{1}{\lambda_1}$. Motivated by this fact, the authors of [13] determine graphs with minimal λ_1 among graphs with given numbers of vertices and edges, and having a given diameter.

Web search engines are based on eigenvectors of the adjacency and some related graph matrices [3, 10].

The indexing structure of object appearing in computer vision (and in a wide range of other domains such as linguistics and computational biology) may take the form of a tree. An indexing mechanism that maps the structure of a tree into a low-dimensional vector space using graph eigenvalues is developed in [12].

3. Energy of $K_n^{(m)}$, $m \geq 2, n \geq 2$

Let G be $K_n^{(m)}$, $m \geq 2, n \geq 2$. The adjacency matrix A of the complete graph K_{n-1} is a square matrix of order $n-1$ given by

$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

Hence the adjacency matrix A_G of G is given by

$$A_G = \begin{pmatrix} 0 & \mathbf{1}_R & \mathbf{1}_R & \cdots & \mathbf{1}_R \\ \mathbf{1}_C & A & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1}_C & \mathbf{0} & A & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_C & \mathbf{0} & \mathbf{0} & \cdots & A \end{pmatrix}$$

Clearly A_G is a square matrix of order $((n-1)m+1)$ with leading term 0; $\mathbf{1}_R$ is a row vector of $n-1$ number of 1's; $\mathbf{1}_C$ is the transpose of $\mathbf{1}_R$; the matrix A on the leading diagonal is the adjacency matrix of K_{n-1} and $\mathbf{0}$ is the zero matrix of order $n-1$.

Theorem 3.1. *Let G be $K_n^{(m)}$, $m \geq 2$, $n \geq 2$. Then*

$$spec(G) = \begin{pmatrix} \frac{(n-2)-\sqrt{n^2+4(m-1)(n-1)}}{2} & \frac{(n-2)+\sqrt{n^2+4(m-1)(n-1)}}{2} & -1 & (n-2) \\ 1 & 1 & m(n-2) & (m-1) \end{pmatrix}$$

and $E(G) = \sqrt{n^2+4(m-1)(n-1)} + (n-2)(2m-1)$.

Proof. The characteristic polynomial of G is

$$|A_G - \lambda I| = \begin{vmatrix} -\lambda & \mathbf{1}_R & \mathbf{1}_R & \cdots & \mathbf{1}_R \\ \mathbf{1}_C & A-\lambda I & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1}_C & \mathbf{0} & A-\lambda I & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_C & \mathbf{0} & \mathbf{0} & \cdots & A-\lambda I \end{vmatrix}$$

with $\mathbf{1}_R$, $\mathbf{1}_C$, A and $\mathbf{0}$ as defined earlier. We name the rows and columns of $A_G - \lambda I$ as R_1, R_i^t, C_1, C_i^t, for $1 \leq t \leq m$ and $1 \leq i \leq n-1$.

Step 1: For every t, $1 \leq t \leq m$, replace R_i^t by $R_i^t - R_1^t$ for $2 \leq i \leq n-1$. Then

$$|A_G - \lambda I| = (1+\lambda)^{m(n-2)} \begin{vmatrix} -\lambda & \mathbf{1}_R & \mathbf{1}_R & \cdots & \mathbf{1}_R \\ \mathbf{1}_C' & A_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1}_C' & \mathbf{0} & A_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_C' & \mathbf{0} & \mathbf{0} & \cdots & A_1 \end{vmatrix}$$

where $\mathbf{1}_C'$ is a column vector of $n-1$ entries given by

$$\mathbf{1}_C' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and A_1 is a matrix of order $n-1$ given by

$$A_1 = \begin{pmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

Step 2: For every t, $1 \leq t \leq m$, replace R_1^t by $\sum_{i=1}^{n-1} R_i^t$. Then the characteristic polynomial of G takes the form

$$|A_G - \lambda I| = (1+\lambda)^{m(n-2)} \begin{vmatrix} -\lambda & \mathbf{1}_R & \mathbf{1}_R & \cdots & \mathbf{1}_R \\ \mathbf{1}_C' & A_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1}_C' & \mathbf{0} & A_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_C' & \mathbf{0} & \mathbf{0} & \cdots & A_2 \end{vmatrix}$$

where

$$A_2 = \begin{vmatrix} -\lambda+(n-2) & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

Step 3: For every t, $2 \leq t \leq m$, replace R_1^t by $R_1^t - R_1^1$.

$$|A_G - \lambda I| = (1+\lambda)^{m(n-2)}(-\lambda+(n-2))^{m-1} \begin{vmatrix} -\lambda & \mathbf{1}_R & \mathbf{1}_R & \cdots & \mathbf{1}_R \\ \mathbf{1}_C^{'} & A_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0}_C & A_2^{''} & A_2^{'} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_C & A_2^{''} & \mathbf{0} & \cdots & A_2^{'} \end{vmatrix}$$

where

$$A_2^{'} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

and

$$A_2^{''} = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Step 4: Replace C_1^1 by $\sum_{t=1}^{m}\sum_{i=1}^{n-1} C_i^t$. Then

$$|A_G - \lambda I| = (1+\lambda)^{m(n-2)}(-\lambda+(n-2))^{m-1} \begin{vmatrix} -\lambda & \mathbf{1}_R^{'} & \mathbf{1}_R & \cdots & \mathbf{1}_R \\ \mathbf{1}_C^{'} & B & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0}_C & \mathbf{0} & A_2^{'} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_C & \mathbf{0} & \mathbf{0} & \cdots & A_2^{'} \end{vmatrix}$$

where

$$B = \begin{pmatrix} -\lambda+(n-2) & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$\mathbf{1}_C^{'} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{0}_C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\mathbf{1}_R^{'} = \begin{pmatrix} m(n-1) & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Step 5: For every t, $2 \leq t \leq m$, replace C_1^t by $\sum_{i=1}^{n-1} C_i^t$. Then

$$|A_G - \lambda I| = (1+\lambda)^{m(n-2)}(-\lambda + (n-2))^{m-1} \begin{vmatrix} -\lambda & \mathbf{1}_R^{'} & \mathbf{1}_R^{''} & \cdots & \mathbf{1}_R^{''} \\ \mathbf{1}_C^{'} & B & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0}_C & \mathbf{0} & A_3^{'} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_C & \mathbf{0} & \mathbf{0} & \cdots & A_3^{'} \end{vmatrix}$$

where

$$\mathbf{1}_R^{''} = \begin{pmatrix} (n-1) & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and

$$A_3^{'} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

Hence

$$|A_G - \lambda I| = (1+\lambda)^{m(n-2)}(-\lambda + (n-2))^{m-1} \begin{vmatrix} -\lambda & m(n-1) \\ 1 & -\lambda + (n-2) \end{vmatrix}$$

$$= (1+\lambda)^{m(n-2)}(-\lambda + (n-2))^{m-1}(\lambda^2 - (n-2)\lambda - m(n-1))$$

Hence

$$spec(G) = \begin{pmatrix} \frac{(n-2)-\sqrt{n^2+4(m-1)(n-1)}}{2} & \frac{(n-2)+\sqrt{n^2+4(m-1)(n-1)}}{2} & -1 & (n-2) \\ 1 & 1 & m(n-2) & (m-1) \end{pmatrix}$$

and

$E(G) = \sqrt{n^2 + 4(m-1)(n-1)} + (n-2)(2m-1).$ □

4. Laplacian, Q-Laplacian, Seidel and Maximum degree Energies of $K_n^{(m)}$, $m \geq 2$, $n \geq 2$

In this section we compute the Laplacian, Q-Laplacian, Seidel and maximum degree energies from the corresponding spectra denoted by $LSpec(G)$, $QSpec(G)$, $SSpec(G)$ and $MDSpec(G)$ respectively.

Theorem 4.1. *Let G be $K_n^{(m)}$, $m \geq 2$, $n \geq 2$. Then*

$$LSpec(G) = \begin{pmatrix} n & 1 & mn-m+1 & 0 \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix}$$

and

$$LE(G) = \frac{m^2(2n^2-4n+2)+2m(n-2)+2}{m(n-1)+1}$$

.

Proof. The characteristic polynomial of G is

$$|L_G - \lambda I| = \begin{vmatrix} -\lambda & \mathbf{1}_R & \mathbf{1}_R & \cdots & \mathbf{1}_R \\ \mathbf{1}_C & L-\lambda I & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1}_C & \mathbf{0} & L-\lambda I & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_C & \mathbf{0} & \mathbf{0} & \cdots & L-\lambda I \end{vmatrix}$$

with $\mathbf{1}_R$, $\mathbf{1}_C$, L and $\mathbf{0}$ as defined earlier.

As in Theorem 3.1, we perform the following steps.

Step 1: For every t, $1 \leq t \leq m$, replace R_i^t by $R_i^t - R_1^t$ for $2 \leq i \leq n-1$.

Step 2: For every t, $1 \leq t \leq m$, replace R_1^t by $R_1^t - \sum_{i=2}^{n-1} R_i^t$.

Step 3: For every t, $2 \leq t \leq m$, replace R_1^t by $R_1^t - R_1^1$.

Step 4: Replace C_1^1 by $\sum_{t=1}^{m}\sum_{i=1}^{n-1} C_i^t$.

Step 5: For every t, $2 \leq t \leq m$, replace C_1^t by $\sum_{i=1}^{n-1} C_i^t$.

These steps give

$$|L_G - \lambda I| = (\lambda - n)^{m(n-2)}(\lambda-1)^{m-1}(\lambda^2 - \lambda(mn-m+1))$$

Consequently

$$LSpec(G) = \begin{pmatrix} n & 1 & mn-m+1 & 0 \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix}$$

and

$$LE(G) = \frac{m^2(2n^2-4n+2)+2m(n-2)+2}{m(n-1)+1}$$

. □

Theorem 4.2. *Let G be $K_n^{(m)}$, $m \geq 2$, $n \geq 2$. Then*

$$QSpec(G) = \begin{pmatrix} n-2 & 2n-3 & \alpha & \beta \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix}$$

where $\alpha = \frac{(mn-m+2n-3)+\sqrt{(mn-m+2n-3)^2-4m(n-1)(2n-4)}}{2}$,
$\beta = \frac{(mn-m+2n-3)-\sqrt{(mn-m+2n-3)^2-4m(n-1)(2n-4)}}{2}$ *and*

$$QE(G) = \frac{m^2(3n^2-10n+7)-2m(n^2-5n+5)-2n+3}{m(n-1)+1} + \sqrt{(mn-m+2n-3)^2-4m(n-1)(2n-4)}$$

for $2 \leq n \leq 4$ *and*

$$QE(G) = \frac{m^2(3n^2-10n+7)-2m(n^2-6n+5)-2n+3}{m(n-1)+1} + m(n-1)+2n-3$$

for $n \geq 5$.

Theorem 4.3. *Let* G *be* $K_n^{(m)}$, $m \geq 2$, $n \geq 2$. *Then*

$$SSpec(G) = \begin{pmatrix} 1 & -(2n-3) & \gamma & \delta \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix}$$

where $\gamma = \frac{(mn-m-2n+3)+\sqrt{(mn-m-2n+3)^2+4m(n-1)}}{2}$,

$\delta = \frac{(mn-m-2n+3)-\sqrt{(mn-m-2n+3)^2+4m(n-1)}}{2}$
and

$$SE(G) = (2mn - m - 2n + 3) + \sqrt{(mn - m - 2n + 3)^2 + 4m(n-1)}$$

Theorem 4.4. *Let* G *be* $K_n^{(m)}$, $m \geq 2$, $n \geq 2$. *Then*

$$MDSpec(G) = \begin{pmatrix} \alpha & \beta & -(n-1) \\ & & \\ 1 & 1 & m(n-1)-1 \end{pmatrix}$$

where $\alpha = \frac{(n-1)(mn-m-1)-\sqrt{((n-1)(mn-m-1))^2+4(m(n-1))^3}}{2}$

and $\beta = \frac{(n-1)(mn-m-1)+\sqrt{((n-1)(mn-m-1))^2+4(m(n-1))^3}}{2}$

and $MDE(G) = (m(n-1)-1)(n-1)+\sqrt{((n-1)(mn-m-1))^2 + 4(m(n-1))^3}$.

5. Conclusion

In this paper we have determined five types of energies for the one-point union of complete graphs on n vertices, $n \geq 2$. In the mathematical and mathematico-chemical literature there exist countless graph matrices; that is matrices defined in terms of certain structural details of the underlying graph. The problem can be studied for such matrices. Computation of Randić energy [2, 11] and sum-connectivity energy [15] are in progress for the graph considered in this paper.

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